Exercise 72

An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \le \theta \le \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

Solution

The domain of the function is $0 \le \theta \le \pi/2$. Take the derivative.

$$\begin{aligned}
F'(\theta) &= \frac{d}{d\theta}\left(\frac{\mu W}{\mu \sin\theta + \cos\theta}\right) \\
&= \frac{\left[\frac{d}{d\theta}(\mu W)\right](\mu\sin\theta + \cos\theta) - \left[\frac{d}{d\theta}(\mu\sin\theta + \cos\theta)\right](\mu W)}{(\mu\sin\theta + \cos\theta)^2} \\
&= \frac{(0)(\mu\sin\theta + \cos\theta) - (\mu\cos\theta - \sin\theta)(\mu W)}{(\mu\sin\theta + \cos\theta)^2} \\
&= -\frac{\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}
\end{aligned}$$

Set $F'(\theta) = 0$ and solve for θ.

$$-\frac{\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = 0$$

$$\mu W(\mu\cos\theta - \sin\theta) = 0$$

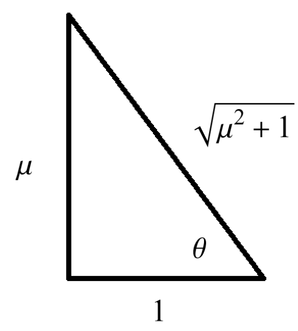
$$\mu\cos\theta - \sin\theta = 0$$

$$\mu\cos\theta = \sin\theta$$

$$\mu = \tan\theta$$

$$\theta = \tan^{-1}\mu$$

μ is a positive number, so $\theta = \tan^{-1}\mu$ is within the interval $0 \le \theta \le \pi/2$. In order to evaluate the function here, draw the implied right triangle.



$$\sin\theta = \frac{\mu}{\sqrt{\mu^2+1}} \qquad \cos\theta = \frac{1}{\sqrt{\mu^2+1}}$$

Evaluate the function at $\theta = \tan^{-1}\mu$.

$$F(\tan^{-1}\mu) = \frac{\mu W}{\mu\left(\frac{\mu}{\sqrt{\mu^2+1}}\right) + \left(\frac{1}{\sqrt{\mu^2+1}}\right)} = \frac{\mu W}{\frac{\mu^2+1}{\sqrt{\mu^2+1}}} = \frac{\mu W}{\sqrt{\mu^2+1}} \qquad \text{(absolute minimum)}$$

Evaluate the function at the endpoints.

$$F(0) = \frac{\mu W}{\mu\sin(0) + \cos(0)} = \mu W$$

$$F\left(\frac{\pi}{2}\right) = \frac{\mu W}{\mu\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = W \qquad \text{(absolute maximum)}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $0 \le \theta \le \pi/2$. Therefore, F is minimized when $\theta = \tan^{-1}\mu$, that is,

$$\tan\theta = \mu.$$